

The diamond rule for multi-loop Feynman diagrams

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Abstract

An important aspect of improving perturbative predictions in high energy physics is efficiently reducing dimensionally regularised Feynman integrals through integration by parts (IBP) relations. The well-known triangle rule has been used to achieve simple reduction schemes. In this work we introduce an extensible, multi-loop version of the triangle rule, which we refer to as the diamond rule. Such a structure appears frequently in higher-loop calculations. We derive an explicit solution for the recursion, which prevents spurious poles in intermediate steps of the computations. Applications for massless propagator type diagrams at three, four, and five loops are discussed.

Keywords: Feynman integrals, integration by parts identities

1. Introduction

Reducing complexities of Feynman integrals through integration by parts (IBP) relations [1, 2] is an important component of modern multi-loop calculations. Finding more efficient reduction methods allows the computation of higher order terms in perturbative expansions which in turn aids in providing a better quantitative understanding of ongoing experiments. Since the 1980s, the so-called *triangle rule* [1, 2] has been used for removing a propagator line from diagrams corresponding to a certain class of integrals. Any topology that has the following substructure can be simplified using the triangle rule:

$$F(a_1, a_2, b, c_1, c_2) = \int d^D k \frac{k^{\mu_1} \dots k^{\mu_N}}{[(k + p_1)^2 + m_1^2]^{a_1} [(k + p_2)^2 + m_2^2]^{a_2} (k^2)^b (p_1^2 + m_1^2)^{c_1} (p_2^2 + m_2^2)^{c_2}}, \quad (1)$$

where D is the dimension which is set to $4 - 2\epsilon$ [3, 4], and b, c_1, c_2 are positive integers. The diagram corresponding to this integral is shown in Fig. 1.

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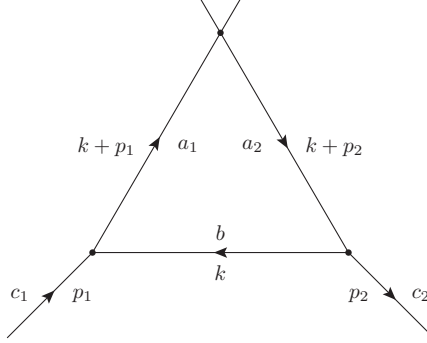


Figure 1: A triangle subtopology where the loop momentum k is assigned to the central line. p_1 and p_2 are external momenta. a_1 , a_2 , b , c_1 , and c_2 represent the powers of their associated propagators.

We write out the IBP relation $\frac{\partial}{\partial k_\mu} k_\mu F = 0$ (where the derivative must be performed before the integration) to obtain

$$1 = \frac{1}{D + N - a_1 - a_2 - 2b} \left[a_1 \mathbf{A}_1^+ (\mathbf{B}^- - \mathbf{C}_1^-) + a_2 \mathbf{A}_2^+ (\mathbf{B}^- - \mathbf{C}_2^-) \right], \quad (2)$$

where \mathbf{A}_i^+ , \mathbf{B}^- , and \mathbf{C}_i^- are operators acting on an integral that increase the power a_i by one, decrease the power b by one, and decrease the power c_i by one, respectively. Numerators that are expressed in dot products of k and an external line, contribute as a constant N to the rule. The rule of the triangle can be recursively applied to remove one of the propagators associated with k , p_1 , or p_2 from the system.

The recursion in the triangle rule can be explicitly ‘solved’ [5], such that the solution is expressed as a linear combination of integrals for which either b , c_1 , or c_2 is 0. The advantage of the summed system over the recursion is that it generates fewer intermediate terms and it cannot have *spurious poles*: terms in which the factor $D + N - a_1 - a_2 - 2b$ becomes proportional to ϵ more than once during the full recursion.

In this work we introduce a more general class of diagrams that can be reduced using an extension of the triangle rule. We call this rule the *diamond rule*. We will show that the diamond rule 1) can be extended to any number of loops, 2) allows for a complete set of irreducible numerators that only contributes as a constant and 3) can be explicitly ‘solved’ to prevent spurious poles.

It should be noted that throughout this paper we mean by reduction the removal of a single line. This does not necessarily mean that the remaining diagram(s) will be trivial. An example of this is the four-loop ladder diagram, which can be reduced twice with the rule of the triangle, after which one of the remaining diagrams involves a master integral and needs a complete reduction scheme of 14 steps in which all numerators are removed and the power of all denominators is lowered to one successively.

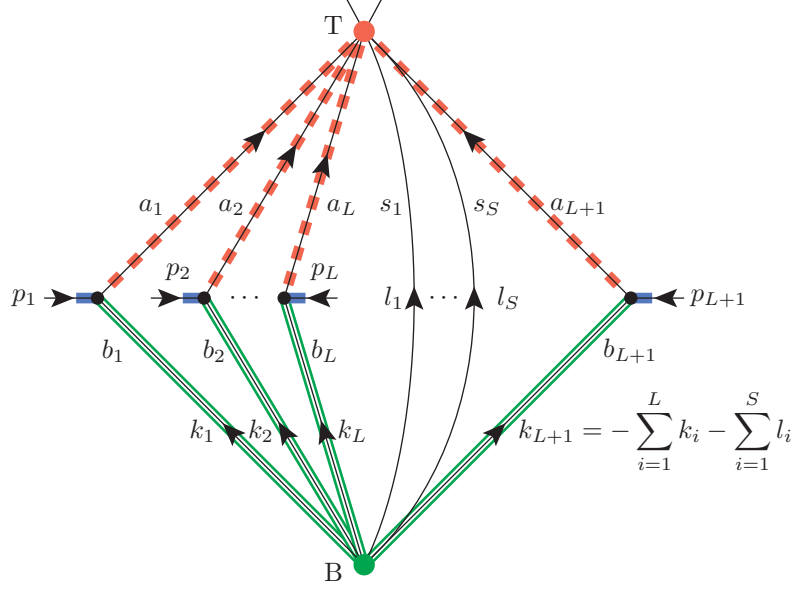


Figure 2: $(L + S)$ -loop diamond-shaped diagram. $(L + 1)$ -lines have external connections and S -lines do not. Red with dashed lines, green with double lines, and blue with thick lines represent upper, lower, and external lines of the diamond, respectively. Label T represents the top vertex, and B the bottom vertex. k_i , p_i , and l_i are momenta, and a_i , b_i , and s_i are the powers of their associated propagators.

In Section 2 the diamond rule is derived. Section 3 shows the explicit summation formula and Section 4 shows examples. Finally, Section 5 gives a conclusion and discussion.

2. Diamond rule

Consider the following family of Feynman integrals in D -dimensions arising from the $(L + S)$ -loop diagram in Fig. 2:

$$F(\{a_i\}, \{b_i\}) = \left[\prod_{i=1}^L \int d^D k_i \right] \left[\prod_{i=1}^S \int d^D l_i \right] \times \left[\prod_{i=1}^{L+1} \frac{k_i^{\mu_1^{(i)}} \dots k_i^{\mu_{N_i}^{(i)}}}{[(k_i + p_i)^2 + m_i^2]^{a_i} (k_i^2)^{b_i}} \right] \left[\prod_{i=1}^S \frac{l_i^{\nu_1^{(i)}} \dots l_i^{\nu_{R_i}^{(i)}}}{(l_i^2)^{s_i}} \right]. \quad (3)$$

The diagram consists of $(L + 1)$ paths from the top vertex T to the bottom vertex B with an external connection in between, and S lines without external connections. The upper, lower, and external lines of the diamond are represented

by red with dashed lines, green with double lines, and blue with thick lines, respectively. The lines without external connections, we call *spectator lines*. In principle any pair of spectator lines can be seen as a two point function which can be reduced to a single line by integration. This line would then have a power that is not an integer. Depending on the complete framework of the reductions this may or may not be desirable. Hence we leave the number of spectators arbitrary. In any case, the contribution of the spectators is a constant (see below), which allows us to characterise integrals in the family only by $2(L+1)$ indices a_i and b_i , and not by s_i . Without loss of generality, we assign loop momenta k_i to the lower lines of the diamond as well as l_i to the spectator lines, except the last diamond line which is fixed by momentum conservation:

$$k_{L+1} = -\sum_{i=1}^L k_i - \sum_{i=1}^S l_i. \quad (4)$$

In contrast, we do not require any constraints on the momentum conservation at the top vertex in the arguments below, hence any number of external lines can be attached to this point. In the middle of the diamond, external lines with momentum p_i are attached by three-point vertices. The upper lines in the diamond may have masses m_i , whereas the lower lines in the diamond and the spectator lines have to be massless. In addition, we allow arbitrary tensor structures of k_i and l_i with homogeneous degrees N_i and R_i , respectively, in the numerator.

Constructing the IBP identity corresponding to the operator

$$\sum_{i=1}^L \frac{\partial}{\partial k_i} \cdot k_i + \sum_{i=1}^S \frac{\partial}{\partial l_i} \cdot l_i, \quad (5)$$

straightforwardly gives the following operator identity:

$$(L+S)D + \sum_{i=1}^{L+1} (N_i - a_i - 2b_i) + \sum_{i=1}^S (R_i - 2s_i) = \sum_{i=1}^{L+1} a_i \mathbf{A}_i^+ [\mathbf{B}_i^- - (p_i^2 + m_i^2)]. \quad (6)$$

Here \mathbf{A}_i^+ and \mathbf{B}_i^- are understood as operators increasing a_i and decreasing b_i by one, respectively, when acting on $F(\{a_i\}, \{b_i\})$. Note that operators changing the spectator indices s_i are absent in the identity.

For a typical usage of Eq. (6), one may identify a diamond structure as a subgraph in a larger graph. If the line with the momentum p_i has the same mass m_i as the corresponding upper line, the term $(p_i^2 + m_i^2)$ in the identity reads as an operator \mathbf{C}_i^- decreasing the corresponding index c_i of the power of the propagator $(p_i^2 + m_i^2)^{-c_i}$ in the larger graph by one. Applying the rule

$$1 = \frac{1}{E} \sum_{i=1}^{L+1} a_i \mathbf{A}_i^+ (\mathbf{B}_i^- - \mathbf{C}_i^-), \quad (7)$$

where

$$E = (L + S)D + \sum_{i=1}^{L+1} (N_i - a_i - 2b_i) + \sum_{i=1}^S (R_i - 2s_i), \quad (8)$$

decreases $\sum_{i=1}^{L+1} (b_i + c_i)$ of integrals appearing in the right-hand side, at the cost of increasing $\sum_{i=1}^{L+1} a_i$. Starting from positive integer indices b_i and c_i , one can repeatedly use the rule until one of either b_i or c_i is reduced to zero.¹

The above diamond rule contains the conventional triangle rule as a special case. For the one-loop case $L = 1$ and $S = 0$, the two lower lines may be identified as a single line and the triangle integral in Eq. (1) can be reproduced. Correspondingly, the IBP identity (7) becomes Eq. (2).

3. Summation rule

We now derive an explicit summation formula for the recursion in the diamond rule. First, we consider the possible connectivities. If we allow for some external lines to be directly connected to each other, we get at least one triangle that can be used for the triangle rule: suppose the external momenta of k_i and of k_j are connected and identified with p_{ij} , then this triangle is k_i, k_j, p_{ij} . In this case, the triangle rule generates fewer terms and is preferred to the diamond rule. Thus, we only consider the case where the diamond does not have direct connections of external lines.

We follow the same procedure as outlined in [5]. First, we rewrite Eq. (7) as:

$$F = \left[\sum_{i=1}^{L+1} a_i \mathbf{A}_i^+ (\mathbf{B}_i^- - \mathbf{C}_i^-) \right] \mathbf{E}^{-1} F, \quad (9)$$

where \mathbf{E} is the operator $(L + S)D + \sum_{i=1}^{L+1} (N_i - a_i - 2b_i) + \sum_{i=1}^S (R_i - 2s_i)$. We split our solution in two classes $\mathbf{A}_i^+ \mathbf{B}_i^-$ and $\mathbf{A}_i^+ \mathbf{C}_i^-$ satisfying

$$\mathbf{E}^{-1}(\mathbf{A}_i^+ \mathbf{B}_i^-) = (\mathbf{A}_i^+ \mathbf{B}_i^-)(\mathbf{E} + 1)^{-1}, \quad \mathbf{E}^{-1}(\mathbf{A}_i^+ \mathbf{C}_i^-) = (\mathbf{A}_i^+ \mathbf{C}_i^-)(\mathbf{E} - 1)^{-1}. \quad (10)$$

We identify the first class with the label $+$, since it increases E by 1, and the latter with the label $-$, since it decreases E by 1. The remaining part of the derivation is analogous to the one in [5].

¹ Note that a_i are allowed to be non-integers provided the denominator in the right-hand side of Eq. (7) never vanishes.

Finally, we obtain the explicit summation formula:

$$\begin{aligned}
F(\{a_i\}, \{b_i\}, \{c_i\}) = & \sum_{r=1}^{L+1} \left[\left(\prod_{\substack{i=1 \\ i \neq r}}^{L+1} \sum_{k_i^+=0}^{b_i-1} \right) \left(\prod_{i=1}^{L+1} \sum_{k_i^-=0}^{c_i-1} \right) (-1)^{k^-} \frac{k_r^+ (k^+ + k^- - 1)!}{\prod_{i=1}^{L+1} k_i^+! k_i^-!} (E + k^+)_{-k^+ - k^-} \right. \\
& \times \left. \left(\prod_{i=1}^{L+1} (a_i)_{k_i^+ + k_i^-} \right) F(\{a_i + k_i^+ + k_i^-\}, \{b_i - k_i^+\}, \{c_i - k_i^-\}) \right]_{k_r^+ = b_r} \\
& + \sum_{r=1}^{L+1} \left[\left(\prod_{i=1}^{L+1} \sum_{k_i^+=0}^{b_i-1} \right) \left(\prod_{\substack{i=1 \\ i \neq r}}^{L+1} \sum_{k_i^-=0}^{c_i-1} \right) (-1)^{k^-} \frac{k_r^- (k^+ + k^- - 1)!}{\prod_{i=1}^{L+1} k_i^+! k_i^-!} (E + k^+ + 1)_{-k^+ - k^-} \right. \\
& \times \left. \left(\prod_{i=1}^{L+1} (a_i)_{k_i^+ + k_i^-} \right) F(\{a_i + k_i^+ + k_i^-\}, \{b_i - k_i^+\}, \{c_i - k_i^-\}) \right]_{k_r^- = c_r}, \tag{11}
\end{aligned}$$

where $k^+ = \sum_{i=1}^L k_i^+$, $k^- = \sum_{i=1}^L k_i^-$, and $(a)_b$ is the rising Pochhammer symbol $\Gamma(a+b)/\Gamma(a)$. The first term decreases the power b_r to 0, and the second term decreases c_r to 0. The only significant difference between the two terms is the +1 in the Pochhammer symbol.

Because the Pochhammer symbol that depends on E only appears once in each term, powers of $1/\epsilon^2$ or higher cannot occur. Thus, the explicit summation formula for the diamond rule does not have spurious poles.

4. Examples

Several examples of diamond structures are displayed in Fig. 3. The role of each line in the diamond rule is highlighted by different colors and shapes. Red dashed lines, green double lines, and blue thick lines represent upper, lower, and external lines of the diamond, respectively. Label T represents the top vertex, and B the bottom vertex. In Fig. 3a a four-loop diagram is displayed. For this diagram, the line of either p_5 , p_6 , p_7 , p_8 , p_9 , or p_{10} can be removed by recursive use of the diamond rule or by the explicit formula given in the previous section. The irreducible numerators of this diagram are selected as $Q \cdot p_8$, $Q \cdot p_{10}$, $p_5 \cdot p_{10}$, and $p_5 \cdot p_7$, such that they adhere to the tensorial structure in the diamond rule. The last numerator, $p_5 \cdot p_7$, lies outside of the diamond and does not interfere with the rule.

If, in this figure, we draw an additional line from the top (T) to the bottom (B) vertex, we obtain the simplest nontrivial propagator topology with a spectator line. As a five-loop diagram it is unique.

In Fig. 3b the three-loop master topology NO is displayed. $Q \cdot p_5$ is chosen as irreducible numerator. One of the lines attached to the diamond is actually an off-shell external line. In general, if the line with momentum p_{L+1} is one of the external momenta of the larger graph, the factor $(p_{L+1}^2 + m_{L+1}^2)$ is just a

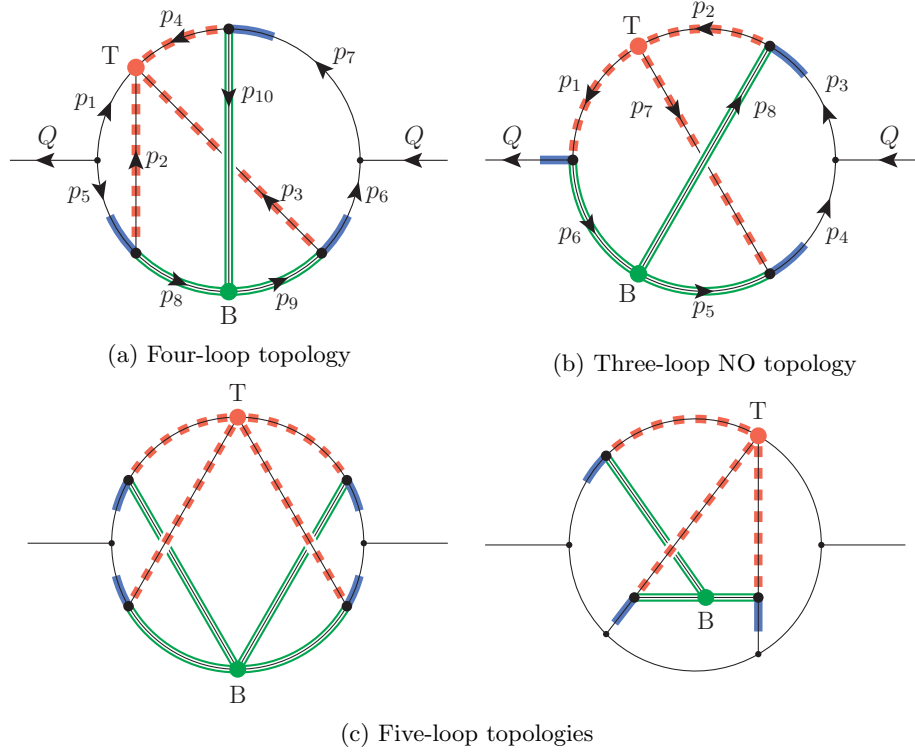


Figure 3: Two topologies with highlighted diamond structures. Red with dashed lines, green with double lines, and blue with thick lines represent upper, lower, and external lines of the diamond, respectively. Label T represents the top vertex, and B the bottom vertex. (a) shows a four-loop topology which can be completely reduced. (b) shows the three-loop NO master topology, for which a modified form of the diamond rule can be applied to lower the power of line p_1 to 1. (c) shows five-loop topologies, which the diamond rule can be applied to.

constant with respect to the loop integration and has no role for reducing the complexity of the integral. As a result, the rule (7) is not applicable to remove one of the internal lines. Even for such cases, one can still find a useful rule by shifting $a_{L+1} \rightarrow a_{L+1} - 1$:

$$1 = \frac{1}{p_{L+1}^2 + m_{L+1}^2} \left[\sum_{i=1}^L \frac{a_i}{a_{L+1} - 1} \mathbf{A}_i^+ \mathbf{A}_{L+1}^- (\mathbf{B}_i^- - \mathbf{C}_i^-) - \frac{E+1}{a_{L+1} - 1} \mathbf{A}_{L+1}^- + \mathbf{B}_{L+1}^- \right], \quad (12)$$

which decreases at least a_{L+1} or b_{L+1} by one. Repeated use of this rule from positive integer a_{L+1} and b_{L+1} reduces a_{L+1} or b_{L+1} to 1. For the NO topology, this variant yields the rule to reduce the line p_1 to 1 in Mincer [6, 7].²

In Fig. 3c we show two five-loop topologies for which the diamond rule can eliminate one line. The first diagram is unique in the sense that it is the simplest diagram for which $L = 3$, $S = 0$. The second diagram is a typical representative of the 29 five-loop topologies with $L = 2$, $S = 0$ and all three p -momenta of the diamond internal.

5. Conclusion and discussion

We have indicated an extensible, multi-loop topology substructure that can be reduced efficiently. We call the corresponding reduction formula the diamond rule. Additionally, we have derived an explicit summation formula for the recursion in the diamond rule, which avoids spurious poles.

For parametric reduction applications such as the Mincer program, an implementation of the diamond rule would be faster than automatically generated reduction rules. This is already the case with the summed triangle as it is used in the Mincer program. It allows the program to avoid spurious poles altogether and hence it can run at a fixed precision in powers of ϵ . It is currently not clear whether Laporta approaches [8] as used in systems such as AIR [9], Reduze [10, 11], and FIRE [12, 13, 14] benefit from applications of the diamond rule.

It is important to note that the tensorial structure for the irreducible numerators should be adhered to. If one chooses numerators of the form $(p_i - p_j)^2$ instead of dot products $p_i \cdot p_j$, extra terms are introduced to the diamond rule. It is unclear to us whether the performance gain of using the rule is greater than the cost of rewriting the numerators to dot products in software such as LiteRed [15, 16]. In the Mincer approach with its dot products in the numerators, the diamond rule fits in perfectly.

Neither at the four-loop level, nor at the five-loop level we have found structures that can be reduced by a single IBP identity, apart from the diamond rule.

²The triangle rule counterpart of this variant was used to reduce the peripheral lines of the massless two-loop propagator-type diagrams with non-integer powers of the central line to unity.

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